10/10/2009 MATH405 Numerical Analysis Name: Solution
Dr. Lunsford Quiz 1 (30 Points Total)

1. Use a classic theorem of calculus to explain why the equation $x - 0.8 = 0.2\sin(x)$ has a solution on the interval $\left[0, \frac{\pi}{2}\right]$. Clearly indicate which theorem you are using and how the assumptions of the theorem are met. NOTE: DO NOT attempt to find the solution! (5 points)

Let $f(x) = x - 0.8 - 0.2\sin x$. Then the equation $x - 0.8 = 0.2\sin x$ has a solution on $[0, \pi/2] \iff \exists\, x_0 \in [0, \pi/2] \ni f(x_0) = 0$. Now, since f is continuous of $[0, \pi/2]$, $f(0) = -0.8 < 0$, $f(\frac{\pi}{2}) = \frac{\pi}{2} - 1 > 0$, and $f(0) \le 0 \le f(\pi/2)$ then by the IVT $\exists\, x_0 \in [0, \pi/2]$ such that $f(x_0) = 0$.

II. Use $p = (13227 - 12832) * 12345$ to answer the following questions. (10 points total)

(a) Compute an approximation to p, say $\hat{p}$, using four digit rounding arithmetic. (4 points)

$\hat{p} = (13230 - 12830) * 12350$

$= 400 \times 12350$

$\boxed{= 4940000}$

Using calculator

$p = 4876275$

(b) Complete the error chart below using your approximation, $\hat{p}$, found in part (a) and your calculator approximation of p as the exact value of p. (3 points)

Absolute Error of Approximation	63725
Relative Error of Approximation	.013068

$|p - \hat{p}|$

$\left|\frac{p - \hat{p}}{p}\right|$

(c) To how many significant digits does $\hat{p}$ approximate p? (3 points)

$.013068 = 1.3068 \times 10^{-2} \le 5 \times 10^{-2}$

$\boxed{\text{2 significant digits}}$

III. Find the 4[th] degree Taylor polynomial, $P_4(x)$, centered at $\dfrac{\pi}{2}$ for the function $\cos x$. DO NOT simplify your answer. (6 points)

Derivs	Eval @ $x=\frac{\pi}{2}$
$f(x)=\cos x$	0
$f'(x)=-\sin x$	-1
$f''(x)=-\cos x$	0
$f^{(3)}(x)=\sin x$	1
$f^{(4)}(x)=\cos x$	0

$$-1\left(x-\tfrac{\pi}{2}\right)+\frac{1}{3!}\left(x-\tfrac{\pi}{2}\right)^3=P_4(x)$$

IV. Use $P_4(x)$ found above to approximate $\cos(100^\circ)$. (4 points)

$$100^\circ=\frac{5\pi}{9}\qquad P_4\left(\tfrac{5\pi}{9}\right)=-1\left(\tfrac{5\pi}{9}-\tfrac{\pi}{2}\right)+\frac{1}{6}\left(\tfrac{5\pi}{9}-\tfrac{\pi}{2}\right)^3$$

$$=-1\left(\tfrac{\pi}{18}\right)+\frac{1}{6}\left(\tfrac{\pi}{18}\right)^3$$

$$\boxed{=-.173646829}$$

V. Use the Taylor remainder, $R_4(x)$, to find an upper bound for the absolute error of the approximation of $\cos(100^\circ)$ using $P_4(x)$. Compare this to the actual absolute error (use your calculator approximation of $\cos(100^\circ)$ as its actual value). (5 points)

$\cos(100^\circ)=\cos\left(\frac{5\pi}{9}\right)=-.1736481777$ ← calc. approx.

$$R_4(x)=\frac{f^{(5)}(\xi(x))}{5!}\left(x-\tfrac{\pi}{2}\right)^5,\quad f^{(5)}(x)=\sin x$$

$$\left|\cos\left(\tfrac{5\pi}{9}\right)-P_4\left(\tfrac{5\pi}{9}\right)\right|=\left|\frac{\sin\left(\xi\left(\frac{5\pi}{9}\right)\right)}{5!}\left(\tfrac{5\pi}{9}-\tfrac{\pi}{2}\right)^5\right|$$

$$\leq\left|\frac{1}{5!}\left(\tfrac{\pi}{18}\right)^5\right|\qquad \text{since } |\sin x|\leq 1$$

$$=1.349601623\times10^{-6}$$ ← upper bound for absolute error

Abs error: $|-.173646829-(-.1736481777)|$

$=1.3487\times10^{-6}$

Abs error is smaller than the upper bound (which it should be) but not by much!

Q: How many significant digits does the approximation in IV have?