3/29/2010 MA371 Intro. to Prob. & Stats. Name: Solution
Dr. Lunsford Quiz 7 (50 Points Total)

Neatly show all work on this quiz. IF you use your calculator to find probabilities for known distributions, then give ALL calculator input including the distribution key you used.

Problem I. A random variable X has density function

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2) & ,-1 \le x \le 1 \\ 0 & ,elsewhere \end{cases}$$

Find the density function of the random variable $Y = X^2$. (10 points)

$0 \le Y \le 1 \quad Y = X^2 \Rightarrow X = \pm\sqrt{Y}$

$$F(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx$$

$$f(y) = \frac{d}{dy}F(y) = f(\sqrt{y})\frac{1}{2\sqrt{y}} - f(-\sqrt{y})\left(\frac{-1}{2\sqrt{y}}\right)$$

$$= \frac{3}{4}(1-y)\frac{1}{2\sqrt{y}} + \frac{3}{4}(1-y)\frac{1}{2\sqrt{y}}$$

$$= \frac{3}{4}(1-y)\frac{1}{\sqrt{y}} \quad 0 < y \le 1,\ f(y) = 0 \text{ elsewhere}$$

Problem II. Show the moment generating function for the exponential distribution is $\frac{1}{1-\theta t}, t < 1/\theta$.

Clearly explain why the restriction on t (i.e. $t < 1/\theta$) is needed. (10 points)

$$m(t) = E[e^{tX}] = \int_0^\infty e^{tx}\frac{1}{\theta}e^{-x/\theta}\,d\theta$$

$$= \frac{1}{\theta}\int_0^\infty e^{x(t-\frac{1}{\theta})}\,dx = \frac{1}{\theta}\frac{1}{t-\frac{1}{\theta}}e^{x\boxed{(t-\frac{1}{\theta})}}\Bigg|_{x=0}^{\infty}$$

needs to be < 0 for integral to converge.

$$= 0 - \frac{1}{\theta(t-\frac{1}{\theta})} = \frac{1}{1-\theta t},\ t < \frac{1}{\theta}$$

If $t - \frac{1}{\theta} < 0$

Problem III. Suppose that customers arrive at a checkout counter at a rate of 2 per minute. Find the following: (5 points each, 20 total)

(a) What is the probability that at least 3 customers will arrive in the next minute? X = # arrivals in next minute

$X \sim \text{Poisson}(\lambda = 2)$

$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2) = 1 - \text{poissoncdf}(2,2)$

$= 0.3233$

(b) A customer has just arrived at the checkout counter. On average, how long will it be before the next customer arrives?

$\frac{2 \text{ arrivals}}{\text{minute}} \Rightarrow \frac{1 \text{ minute}}{2 \text{ arrivals}}$ $\boxed{\frac{1}{2} \text{ minute}}$

(c) A customer has just arrived at the checkout counter. What is the probability that it will be at least 4 minutes before the next customer arrives? Would you classify this event as very likely, somewhat likely, somewhat unlikely, or very unlikely? Why?

X = time until next arrival

X exponential $\theta = \frac{1}{2}$

$P(X \geq 4) = \int_4^{\infty} 2e^{-2x}\,dx = e^{-8}$

$= 0.000335$

X = # arrivals in 4 minutes

$X \sim \text{poisson}(\lambda = 8)$

$P(X = 0) = e^{-8} = 0.000335$

very unlikely (less than 0.1% chance!)

(d) If a clerk takes 3 minutes to serve the first customer arriving at the counter, what is the probability that at least one more customer will be waiting when the service to the first customer is completed?

X exponential

X - time until next arrival, $\theta = \frac{1}{2}$

$P(X \leq 3) = 1 - P(X \geq 3)$

$= 1 - e^{-2(3)} = 1 - e^{-6}$

$= 0.9975$

X = # arrivals in 3 minutes

X poisson $\lambda = 2(3\text{ min}) = 6$

$P(X \geq 1) = 1 - P(X = 0)$

$= 1 - e^{-6} = .9975$

Problem IV. The number of industrial accidents at a particular manufacturing plant is found to average three per month. Please answer the following: (5 points each, 10 points total)

(a) What is the percent chance that the plant will have at least six accidents during a particular month?

X # accident in month

X poisson, $\lambda = 3$

$P(X \geq 6) = 1 - P(X < 6)$

$= 1 - \text{poissoncdf}(3,5)$

$= 0.0839$

(b) During the previous month the plant had six accidents. Does this number seem improbable if the average number of accidents per month is still equal to 3? Does it indicate a possible increase in the mean number of accidents per month? Do you think this warrants an investigation? You should base your answers on your probability computation in part (a).

From part (a) we see the probability of 6 or more accidents in a month is somewhat unlikely (8.4% chance). It could signal an increase in the mean number of accidents per month. I think it warrants some looking into.