Pledge:

4/21/2010 MATH361 Calculus III Name: Solution

Dr. Lunsford Test 2 (100 Points Total)

Please show all work on this test. You may (or may not) find the following formulas useful.

$$\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin(2u) \qquad \int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin(2u) \qquad \iint_S 1\,dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$\int f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt \qquad \int \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C P\,dx + Q\,dy \qquad \oint_C F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Problem I. Find the maximum rate of change of $f(x,y) = \frac{x^2}{y}$ at the point $(4,2)$ and the direction in which it occurs. Clearly indicate your answers. (8 points)

$\nabla f = \langle \frac{2x}{y}, -x^2 y^{-2} \rangle$

$\nabla f(4,2) = \langle 4, -4 \rangle$ Direction

maximum $= |\nabla f(4,2)|$

$= 4\sqrt{2}$

Problem II. Find the directional derivative of $f(x,y) = \ln(x^2+y^2)$ at the point $(2,1)$ in the direction of the vector $\mathbf{v} = \langle -1, 2 \rangle$. (8 points)

$\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$

$\nabla f(2,1) = \langle \frac{4}{5}, \frac{2}{5} \rangle$

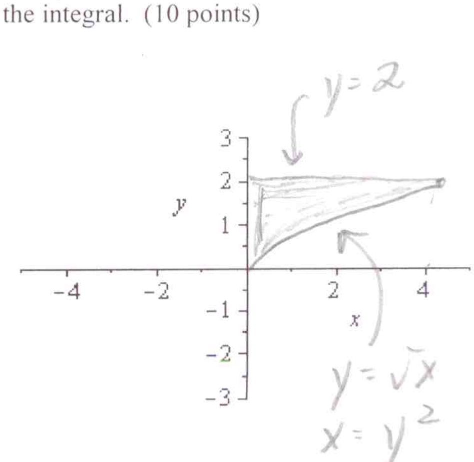
$\nabla f \cdot \frac{\vec{v}}{|\vec{v}|} = \langle \frac{4}{5}, \frac{2}{5} \rangle \cdot \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$= -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$

Problem III. Change the order of integration for the following integral. On the axes provided include a graph of the region R over which the integral is defined. DO NOT EVALUATE the integral. (10 points)

$$\int_0^4 \int_{\sqrt{x}}^2 \cos(y^2)\,dy\,dx$$

$$\int_0^2 \int_0^{y^2} \cos y^2\,dx\,dy$$



Problem IV. Set up, but DO NOT EVALUATE, an integral to find the surface area of the parametric surface given by the vector function $\mathbf{r}(u,v) = v^2\mathbf{i} - uv\mathbf{j} + u^2\mathbf{k}$ for $0 \le u \le 3$ and $-3 \le v \le 3$. (10 points)

$\vec{r}_u = \langle 0, -v, 2u \rangle$

$\vec{r}_v = \langle 2v, -u, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u^2, 4uv, 2v^2 \rangle$

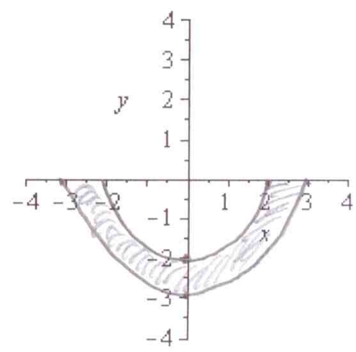
$|\vec{r}_u \times \vec{r}_v| = 2\sqrt{u^2 + 4u^2v^2 + v^2}$

$\iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

$$\int_{-3}^{3}\int_{0}^{3} 2\sqrt{u^2+4u^2v^2+v^2}\, du\, dv$$

Problem V. Change the following integral to polar coordinates: $\iint_R (x+y)\,dA$ where R is the region below the x-axis and between the graphs of $x^2+y^2=4$ and $x^2+y^2=9$. On the axes provided include a graph of the region R over which the integral is defined. DO NOT EVALUATE the integral. (10 points)

$$\int_{\pi}^{2\pi}\int_{2}^{3} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$



Problem VI. Evaluate the line integral $\int_C y\sin(z)\,ds$ where C is the circular helix given by $x=\cos(t)$, $y=\sin(t)$ and $z=t$, $0 \le t \le 2\pi$. (10 points)

$ds = |\vec{r}\,'(t)|\,dt$

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_0^{2\pi} \sin(t)\sin(t)\sqrt{2}\,dt$$

$$\int_0^{2\pi} \sqrt{2}\sin^2 t\, dt$$

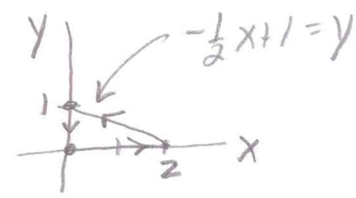
$$= \sqrt{2}\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \sqrt{2}\left(\tfrac{1}{2}(2\pi) - 0\right)$$

$$= \sqrt{2}\pi$$

Problem VII. Use Green's Theorem to evaluate $\int_C \overset{P}{x^4}dx + \overset{Q}{xy}dy$ where C is the triangular curve, oriented counterclockwise, consisting of the line segments from (0,0) to (2,0), from (2,0) to (0,1), and from (0,1) to (0,0). (10 points)

$$\frac{\partial P}{\partial y} = 0, \quad \frac{\partial Q}{\partial x} = y$$



$$\iint_A y\,dA = \int_0^2 \int_0^{1-\frac{1}{2}x} y\,dy\,dx$$

$$= \int_0^2 \frac{1}{2}y^2 \Big|_0^{1-\frac{1}{2}x} = \int_0^2 \frac{1}{2}\left(1-\frac{1}{2}\right)^2 dx = \frac{1}{3}$$

Problem VIII. Let $\mathbf{F} = (3y-2z)\mathbf{i} + (3x+z)\mathbf{j} + (y-2x)\mathbf{k}$. Please answer the following questions: (16 points total)

(a) Find $f(x,y,z)$ such that $\mathbf{F} = \nabla f$. (10 points)

$$f_x = 3y-2z$$
$$f_y = 3x+z$$
$$f_z = y-2x$$

$$f(x,y,z) = \int 3y-2z\,dx = 3yx-2xz+g(y,z)$$
$$f_y(x,y,z) = 3x+g_y(y,z)$$
$$3x+z = 3x+g_y(y,z)$$
$$\Rightarrow g_y(y,z) = z \Rightarrow g(y,z) = yz+h(z)$$

$$\Rightarrow f(x,y,z) = 3yx - 2xz + yz + h(z)$$
$$f_z(x,y,z) = -2x + y + h'(z)$$
$$y-2x = y-2x+h'(z)$$
$$\Rightarrow h'(z) = 0$$
$$\Rightarrow h(z) = K$$

$$\therefore \boxed{f(x,y,z) = 3yx-2xz+yz+K}$$

(b) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the line segment from $\underset{\vec{r}(a)}{(1,2,3)}$ to $\underset{\vec{r}(b)}{(1,4,5)}$. (6 points)

Since $\nabla f = \vec{F}$ use the F.T.L.I.

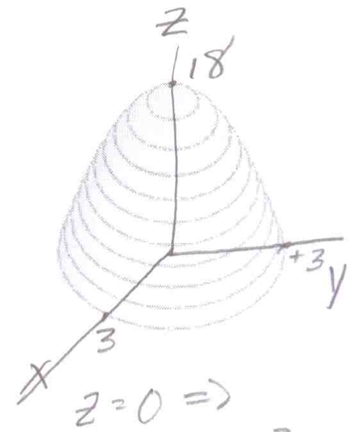
$$\int \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(1,4,5) - f(1,2,3) = 16$$

Note: You could also parameterize the line segment and compute

$$\int_C \vec{F}\cdot d\vec{r}$$

Problem IX. We wish to evaluate the integral $\iiint_B yz\,dV$ where B is the solid region below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane. To your right you are given a graph of the solid with the lines on the solid drawn parallel to the xy-plane. Please answer the following: (12 points total)



$z = 0 \Rightarrow$
$9 = x^2 + y^2$

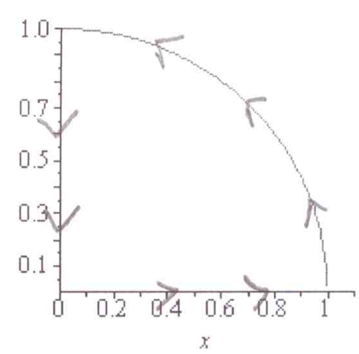
(a) Set up, DO NOT EVALUATE, the integral in rectangular coordinates. (6 points)

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{18-2x^2-2y^2} zy\,dz\,dy\,dx$$

(b) Set up, DO NOT EVALUATE, the integral in cylindrical coordinates. (6 points)

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{18-2r^2} z\,r\sin\theta\,r\,dz\,dr\,d\theta$$

Problem X. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle xy^2, x^2y\rangle$ and C is the path, oriented counterclockwise, consisting of the line segment from (0,0) to (1,0), then along the circular arc $y = \sqrt{1-x^2}$ from (1,0) to (0,1), then along the line segment from (1,0) to (0,0). The graph below may be helpful. Hint: Think conservative! (6 points)



Notice $\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x}$

$\therefore$ F is conservative.

Since C is a simple closed curve and F is conservative the $\int_C \vec{F}\cdot d\vec{r} = 0$

You could also apply Green's theorem directly:

$$\int_C xy^2\,dx + x^2y\,dy = \iint_D (2xy - 2xy)\,dA = \iint_D 0\,dA$$

$= 0.$