4/21/2009 MATH261 Calculus I Name: Solution
Dr. Lunsford Test 2 (100 Points Total)

Problem I. Find the indicated derivatives and integrals. Neatly show all of your work and simplify your answers. Very little partial credit (either all, half, or nothing) will be given for these problems. Clearly indicate your answers: (4 points each, 28 points total)

1. $y = \cos(8x)$, $\frac{dy}{dx} = \boxed{-8\sin(8x)}$

2. $\int 5\cos(4x)\,dx = \boxed{\frac{5}{4}\sin(4x) + C}$

3. $w = \ln(u^3 + u)$, $\frac{dw}{du} = \boxed{\frac{3u^2+1}{u^3+u}}$

4. $\int \frac{3}{1+x^2}\,dx = \boxed{3\arctan x + C}$

5. $y = \frac{1}{4x^{10}}$, $\frac{dy}{dx} = -\frac{10}{4}x^{-11} = \boxed{-\frac{5}{2}x^{-11}}$

$\downarrow$

$\frac{1}{4}x^{-10}$

6. $\int_0^1 2t(t^2+1)^4\,dt = \int_1^2 u^4\,du = \frac{1}{5}u^5\Big|_1^2 = \frac{1}{5}(2)^5 - \frac{1}{5}(1)^5 = \boxed{\frac{31}{5}}$

$u = t^2+1$

$du = 2t\,dt$

$t=0 \Rightarrow u=1,\ t=1 \Rightarrow u=2$

7. $y = \sqrt[3]{1-x}$, $\frac{dy}{dx} = \frac{1}{3}(1-x)^{-2/3}(-1) = \boxed{-\frac{1}{3}(1-x)^{-2/3}}$

$\downarrow$

$(1-x)^{1/3}$

Problem II. Show that $F(x) = x\ln x - x$ is an anti derivative of $f(x) = \ln x$. (4 points)

$F'(x) = 1\ln x + x\left(\frac{1}{x}\right) - 1 = \ln x + 1 - 1 = \ln x$

Since $F'(x) = f(x)$ then F is an antiderivative of f.

Problem III. A particle moves along a straight path with acceleration $a(t) = t - 1$ feet per second squared. At time $t = 1$ second the particle has velocity 2 feet per second and is positioned at 3 feet. Find the position function of the particle. (5 points)

$v(t) = \int a(t)\,dt = \int t - 1\,dt = \frac{1}{2}t^2 - t + C$

$v(1) = 2 \Rightarrow \frac{1}{2} - 1 + C = 2 \Rightarrow C = 5/2 \Rightarrow v(t) = \frac{1}{2}t^2 - t + 5/2$

$p(t) = \int v(t)\,dt = \int \frac{1}{2}t^2 - t + 5/2\,dt = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{5}{2}t + d$

$p(1) = 3 \Rightarrow \frac{1}{6} - \frac{1}{2} + \frac{5}{2} + d = 3 \Rightarrow d = 5/6$

$\therefore \boxed{p(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{5}{2}t + 5/6}$

Problem IV. Find the indicated limits. You must show at least one intermediate step for these limits and you must clearly show all work. (5 points each, 10 points total)

1. $\lim_{x\to 0}\frac{e^{3x}-1}{x} \overset{L'H}{=} \lim_{x\to 0}\frac{3e^{3x}}{1} = \boxed{3}$

$\frac{0}{0}$

2. $\lim_{t\to\infty}\left(t\sin\left(\frac{4}{t}\right)\right) = \lim_{t\to\infty}\frac{\sin\left(\frac{4}{t}\right)}{\frac{1}{t}} \overset{L'H}{=} \lim_{t\to\infty}\frac{\cos\left(\frac{4}{t}\right)\left(-\frac{4}{t^2}\right)}{-\frac{1}{t^2}}$

$\infty \cdot 0$ $\qquad \frac{0}{0}$

$= \lim_{t\to\infty} 4\cos\left(\frac{4}{t}\right) = \boxed{4}$

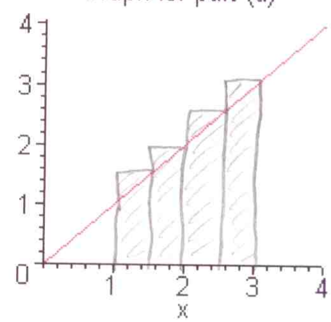
Problem V. Consider the integral $\int_1^3 x\,dx$. Please answer the following: (10 points total)

(a) Estimate the integral using **a right endpoint sum** with 4 subintervals. Clearly show all of your work and use the graph to your right to show the sum graphically. (5 points)

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

Right sum $= \frac{1}{2}\left[1.5+2+2.5+3\right] = \frac{9}{2}$

Graph for part (a)

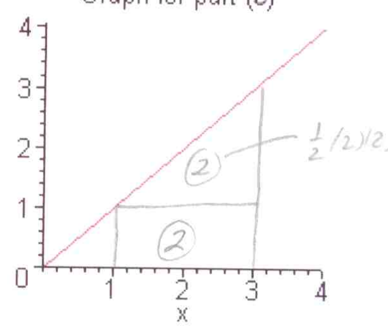


(b) Now find the integral using only geometry. You should indicate how you are doing this on the graph to your right. Check your geometric answer by performing the integration. (5 points)

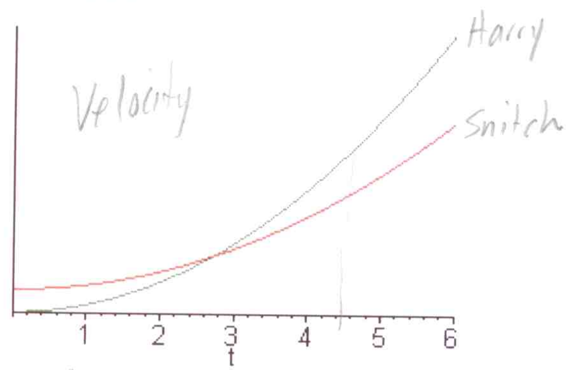
area $\Delta = 2$ area $\square = 2$ $\therefore$ area $= 4$

check: $\int_1^3 x\,dx = \frac{1}{2}x^2\Big|_1^3 = \frac{1}{2}(9-1) = \frac{8}{2} = 4$

Graph for part (b)



Problem VI. At time $t = 0$ the "golden snitch" (an incredibly fast flying object) passes directly Harry Potter's head and Harry takes off after the snitch on his Nimbus 2000 Broomstick (i.e. at time $t = 0$ Harry and the snitch are at the same position). Below you are given a graph of the ***velocity*** functions of Harry and the Snitch. Assume Harry and the snitch are moving along a straight path. At approximately what time will Harry catch up to the snitch? You must justify your answer using the graph to your right to get full credit. (4 points)



Since the velocities are both positive, the distance travelled from $t=0$ to $\hat{t}$ will be the area under each curve from $t=0$ to $\hat{t}$. Those areas are approximately the same at $\hat{t} \approx 4.5$

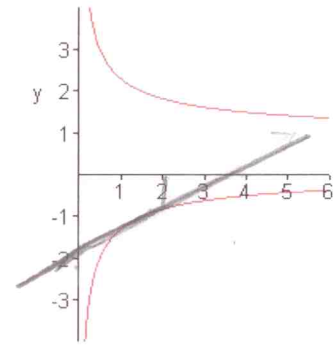
i.e. $\int_0^{4.5}$ Harry vel $dt \approx \int_0^{4.5}$ snitch vel dt.

Problem VII. Below you are given part of the graph of $y^2x - yx^2 = 3$. Find and accurately draw the tangent line to the graph at the point $(2,-1)$. Neatly show all of your work and clearly indicate your answer. Hint: Use implicit differentiation. (6 points)

$$2y\frac{dy}{dx}x + y^2 - \left(\frac{dy}{dx}x^2 + y2x\right) = 0$$

$$\frac{dy}{dx} = \frac{-y^2+2xy}{2xy-x^2}, \quad \frac{dy}{dx}\Big|_{(2,-1)} = \frac{-1-4}{-4-4} = \frac{5}{8}$$

$$y-(-1) = \frac{5}{8}(x-2) \Rightarrow y = \frac{5}{8}x - \frac{9}{4}$$



Problem VIII. More interesting derivatives and integrals. Please find the indicated derivatives. Do not simplify your answers. Clearly indicate your answers. (6 points each – 24 points total)

(a) $y = x^{\sin(4x)}$, $\frac{dy}{dx} = ?$

$$\ln y = \ln\left(x^{\sin(4x)}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(\sin(4x)\ln x\right)$$

$$\frac{dy}{dx} = y\left[4\cos 4x \ln x + \frac{\sin 4x}{x}\right]$$

$$\frac{dy}{dx} = \left(x^{\sin(4x)}\right)\left(4\cos(4x)\ln x + \frac{\sin(4x)}{x}\right)$$

(b) $f(x) = \dfrac{e^{3x}}{(x^2+1)^3}$, $f'(x) = ?$

$$= e^{3x}(x^2+1)^{-3}$$

$$3e^{3x}(x^2+1)^{-3} + e^{3x}\left(-3(x^2+1)^{-4}(2x)\right)$$

OR

$$\frac{3e^{3x}(x^2+1)^3 - e^{3x}\left(3(x^2+1)^2(2x)\right)}{(x^2+1)^6}$$

(c) $\displaystyle\int_1^2 t\sqrt{t-1}\,dt = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 u^{3/2} + u^{1/2}\,du$

$u = t-1 \Rightarrow t = u+1$

$du = dt$

$t=1 \Rightarrow u=0, \; t=2 \Rightarrow u=1$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{5} + \frac{2}{3} = \boxed{\frac{16}{15}}$$

(d) $\displaystyle\int \frac{\sqrt{x}+2x+5}{x^2}\,dx = \int \left(x^{1/2}+2x+5\right)x^{-2}\,dx$

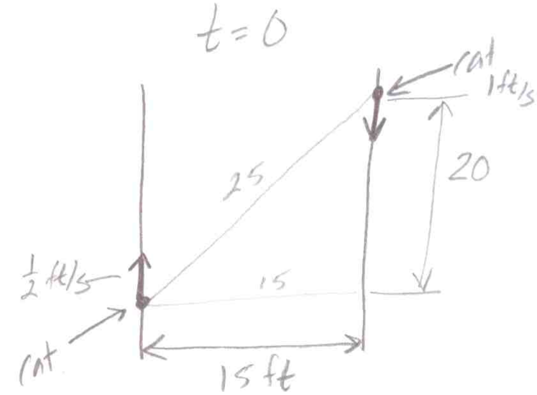
$$= \int x^{-3/2} + 2x^{-1} + 5x^{-2}\,dx = \boxed{-2x^{-1/2} + 2\ln|x| - 5x^{-1} + C}$$

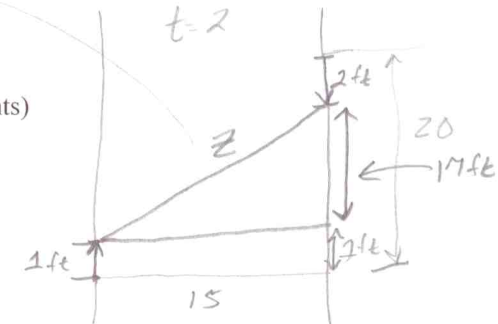
Problem IX. An alley is bounded by two fences that are 15 feet apart. At time $t = 0$ we observe two alley cats slinking towards each other but on different fences (please see the drawing below). One cat is slinking at a rate of $\frac{1}{2}$ foot per second while the other cat is slinking at a rate of 1 foot per second. Please answer the following questions: (9 points total)

(a) If the cats are 25 feet apart at time $t = 0$ seconds, then how far apart are they at $t = 2$ seconds? (3 points)

$z^2 = 15^2 + 17^2$

$z = \sqrt{514} \approx 22.67$ ft.





(b) How fast is the distance between the cats changing at $t = 2$ seconds? (6 points)

Let x be closing distance between cats measured along fence 1 (also note $x = 20 - 1.5t$)

Let z be the distance between cats

Then

$15^2 + x^2 = z^2$

and $\frac{dx}{dt} = -1.5\ ft/s$

so $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$

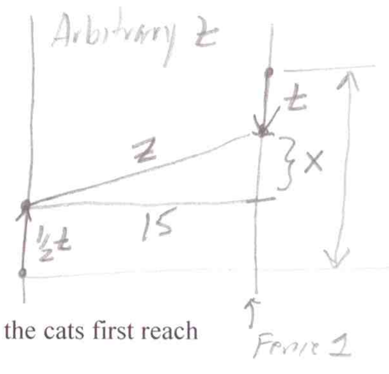
$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$

@ t=2 seconds

$\left.\frac{dz}{dt}\right|_{t=2} = \frac{17}{\sqrt{514}}(-1.5\ ft/s)$

$= -1.125\ ft/s$

$\boxed{\text{Decreasing @ } 1.125\ ft/s}$



BONUS: Hissing distance is reached when the cats are 21 feet apart. At what time will the cats first reach hissing distance? (4 points)

$21^2 = 15^2 + x^2$

$\Rightarrow x = \pm\sqrt{21^2 - 15^2} = \pm\sqrt{216}$

$x = 20 - 1.5t$

$20 - 1.5t = \pm\sqrt{216}$

$t = \frac{20 \pm \sqrt{216}}{1.5}$

1st time:

$\frac{20 - \sqrt{216}}{1.5} = 3.54$ seconds