11/21/2000 MA430 Theory of Prob/Stats Name: <u>SOLUTION</u>
Dr. Lunsford Test 2 (100 Points Total)

Neatly show all work on this test. Clearly indicate your answers. Good luck!

I. Multiple Choice. Circle the best answer for each problem. (3 points each – 15 points total) (Answer is CAPITALIZED)

1. A fair die is tossed 10 times and the outcome of each toss is recorded. Let X be the number of twos and threes that appear in the 10 tosses. Then X has which of the following binomial distributions:
:

(a) $b(10,\frac{1}{6})$ (b) $b(10,\frac{1}{2})$ (C) $b(10,\frac{1}{3})$ (d) $b(6,\frac{1}{6})$

2. If the random variable X has the cumulative distribution function (c.d.f.)

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

then $P(\frac{1}{2} \le X \le \frac{3}{2}) =$

(a) 4/5 (B) 7/8 (c) 26/8 (d) 4 (e) None of these

3. Let X and Y be two discrete random variables defined on a sample space S. Which of the following are equivalent expressions for the p.d.f. $f_{X,Y}(x,y)$?

(i) $P(\{s \in S \mid X(s) = x, Y(s) = y\})$
(ii) $f_X(x) f_Y(y)$
(iii) $P(X = x, Y = y)$

(a) (i) and (ii) (b) (ii) and (iii) (C) (i) and (iii) (d) all three

4. Suppose X is $N(0,1)$ (i.e. X is standard normal). Then
$P(-1.82 \le X \le 1.82) =$

(A) .9312 (b) .9656 (c) .0344 (d) .9296 (e) None of these

5. If X is $N(0,1)$ then the value of c such that $P(|X| < c) = .90$ is

(a) 1.96 (b) –1.96 (c)1.19 (d) 2.575 (e) –2.575 (F) None of these

II. Five cards are dealt from a poker deck (52 cards with 4 suits and 13 denominations in each suit) to form a hand. What is the probability that the hand will be a three of a kind (i.e. the hand will contain exactly 3 cards of the same denomination)? (7 points)

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.021$$

III. Six of the twenty houses on Main Street in Anytown USA have chimneys. If 3 houses on Main Street are chosen at random, what is the probability that two of them will have chimneys? (7 points)

Let X be the number of the 3 houses chosen that have chimneys. Then X is hypergeometric and $P(X=2) = \frac{\binom{6}{2}\binom{14}{1}}{\binom{20}{3}} = \frac{7}{38} = .184$

IV. A biased coin (probability of a Head is .6) is flipped 10 times and the result of each flip is recorded. What is the probability that at least one Tail will appear in the 10 flips? (7 points)

Let X be the number of the Tails that appear in the 10 flips (i.e. think of Tails as a "success" with probability .4). Then X is $b(10,0.4)$. The probability that at least one Tail appears is the compliment of the event that no Tails appear. Thus we have

$$P(X \geq 1) = 1 - P(X=0) = 1 - \binom{10}{0}(.4)^0(.6)^{10} = 1-(.6)^{10} = .99395$$

V. A sample space, S, for an experiment has four equally likely outcomes given below:

$$S = \{(1,2,0),(2,1,0),(0,1,1),(1,0,0)\}$$

Define the random variable Y on the sample space S by $Y(s) = Y((a,b,c)) = a+b+c$ for each s in S. Find the probability density function (p.d.f.) for Y, $f_Y(y)$, and graph it on the axes provided below. (8 points)

Y takes on the values 1, 2, and 3. Since each outcome in the sample space is equally likely, we have:

$$f_Y(1) = P(\{s \in S \mid Y(s) = 1\}) = P(\{(1,0,0)\}) = 1/4$$

$$f_Y(2) = P(\{s \in S \mid Y(s) = 2\}) = P(\{(0,1,1)\}) = 1/4$$
$$f_Y(3) = P(\{s \in S \mid Y(s) = 3\}) = P(\{(1,2,0),(2,1,0)\}) = 1/2$$

Thus $f_Y(y) = \begin{cases} 1/4, & y = 1,2 \\ 1/2, & y = 3 \\ 0, & elsewhere \end{cases}$

VI. Let Y be a random variable with p.d.f. $f_Y(y) = \begin{cases} y, & 0 \le y < 1 \\ 2 - y, & 1 \le y \le 2 \\ 0, & elsewhere \end{cases}$. A graph of $f_Y(y)$ is given below. Find the c.d.f. for Y. (10 points)

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{2}y^2, & 0 \le y < 1 \\ -\frac{1}{2}y^2 + 2y - 1, & 1 \le y < 2 \\ 1, & 2 \le y \end{cases}$$

VII. Let X be the amount of time (in years) spent in jail by persons convicted of grand theft auto. If X has the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{9}x^2, & 0 < x < 3 \\ 0, & elsewhere \end{cases}$$

then what is the average length of time spent in jail for persons convicted of grand theft auto (i.e. what is the expected value of X)? (8 points)

$$E(X) = \int_0^3 x(\frac{1}{9}x^2)\,dx = 2.25 \text{ years}$$

VIII. A random sample of 100 consumers is questioned to determine their soft drink brand preference. Let $X_i = 0$ if the i^{th} consumer questioned prefers brand A, let $X_i = 1$ if the i^{th} consumer questioned prefers brand B, let $X_i = 2$ if the i^{th} consumer questioned prefers brand C, where $i = 1,\ldots,100$. Suppose 33 of the 100 respondents preferred brand A, 50 preferred brand B, and the remaining preferred brand C. What is the sample mean of this random sample? (7 points)

$$\overline{X} = \frac{1}{100}\sum_{i=1}^{100} X_i = \frac{1}{100}(33(0) + 50(1) + 17(2)) = .84$$

X. A random variable has p.d.f. $f_X(x) = \begin{cases} 1/6, & x = 0 \\ 1/3, & x = 1 \\ 1/2, & x = 2 \\ 0, & elsewhere \end{cases}$

Find $Var(X)$. (8 points)

$E[X] = 0(1/6) + 1(1/3) + 2(1/2) = 4/3$

$$Var(X) = E[(X-m)^2] = (0-4/3)^2(1/6) + (1-4/3)^2(1/3) + (2-4/3)^2(1/2) = 5/9$$

OR

$$Var(X) = E(X^2) - m^2 = 0(1/6) + 1(1/3) + 4(1/2) - (4/3)^2 = 5/9$$

X. Let X be a random variable with $E[X] = m$. By using properties of expected value show that $Var(X) = E[X^2] - m^2$. (8 points)

$Var(X) = E[(X-m)^2]$ (by the definition of variance)

$= E[X^2 - 2mX + m^2]$ (by basic algebra)

$= E[X^2] - 2mE[X] + E[m^2]$ (linearity of expected value)

$= E[X^2] - 2m^2 + m^2$ (since $E[X] = m$ and $E[c] = c$ for any constant c)

$= E[X^2] - m^2$ (basic algebra)

XI. Let X be $N(6,25)$. Find each of the following (5 points each – 15 total)

(a) $P(X > 11) = P(\frac{X-6}{5} > \frac{11-6}{5}) = 1 - P(\frac{X-6}{5} \le 1) = .1587$

(b) $P(0 \le X \le 8) = P(\frac{0-6}{5} \le \frac{X-6}{5} \le \frac{8-6}{5}) = P(-1.2 \le \frac{X-6}{5} \le .4) = .5403$

(c) $P(|X-6| \le 5) = P(-5 \le X-6 \le 5) = P(-1 \le \frac{X-6}{5} \le 1) = .6826$