Probability Review and Counting Fundamentals

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Overview

- Probability Review

- Fundamentals of Counting
  - Permutations: *ordered arrangements*
  - Combinations: *unordered arrangements*

- Selected Activities
Probability Review

- Definitions
- Classical Probability
- Relative Frequency Probability
- Probability Fundamentals and Probability Rules
What is Probability?

- Probability
  
  the study of chance associated with the occurrence of events

- Types of Probability
  
  - Classical (Theoretical)
  - Relative Frequency (Experimental)
Rolling dice and tossing a coin are activities associated with a classical approach to probability. In these cases, you can list all the possible outcomes of an experiment and determine the actual probabilities of each outcome.
Listing All Possible Outcomes of a Probabilistic Experiment

- There are various ways to list all possible outcomes of an experiment
  - Enumeration
  - Tree diagrams
  - Additional methods – counting fundamentals
Three Children Example

- A couple wants to have exactly 3 children. Assume that each child is either a boy or a girl and that each is a single birth.
- List all possible orderings for the three children.
<table>
<thead>
<tr>
<th>1&lt;sup&gt;st&lt;/sup&gt; Child</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Child</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; Child</th>
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</tbody>
</table>
**Enumeration**

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<thead>
<tr>
<th>1&lt;sup&gt;st&lt;/sup&gt; Child</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Child</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; Child</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
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<tr>
<td>G</td>
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<td>B</td>
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<tr>
<td>G</td>
<td>G</td>
<td>G</td>
</tr>
</tbody>
</table>
Tree Diagrams

1st Child  2nd Child  3rd Child

B → B → B

B → G → G

G → B → G

G → G → G
Definitions

- **Sample Space** - the list of all possible outcomes from a probabilistic experiment.
  - 3-Children Example: 
    \[ S = \{\text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}\} \]
  - Each individual item in the list is called a **Simple Event** or **Single Event**.
Probability Notation

\[ P(\text{event}) = \text{Probability of the event occurring} \]

Example: \( P(\text{Boy}) = P(B) = \frac{1}{2} \)
Probability of Single Events with Equally Likely Outcomes

- If each outcome in the sample space is **equally likely**, then the probability of any one outcome is 1 divided by the total number of outcomes.

For equally likely outcomes,

\[ P(\text{simple event}) = \frac{1}{\text{total number of outcomes}} \]
A couple wants 3 children. Assume the chance of a boy or girl is \textit{equally likely} at each birth.

What is the \textit{probability} that they will have \textbf{exactly} 3 girls?

What is the \textit{probability} of having \textbf{exactly} 3 boys?
Probability of Combinations of Single Events

- An event can be a combination of *Single Events*.

- The probability of such an event is the sum of the individual probabilities.
Three Children Example
Continued

$P(\text{exactly 2 girls}) = \_

P(\text{exactly 2 boys}) = \_

P(\text{at least 2 boys}) = \_

P(\text{at most 2 boys}) = \_

P(\text{at least 1 girl}) = \_

P(\text{at most 1 girl}) = \_

\begin{itemize}
  \item Sample space = \_
\end{itemize}
Types of Probability

- Classical (Theoretical)
- Relative Frequency (Experimental, Empirical)
Relative Frequency Probability

- Uses actual experience to determine the likelihood of an outcome.
- What is the chance of making a B or better?

<table>
<thead>
<tr>
<th>Grade</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20</td>
</tr>
<tr>
<td>B</td>
<td>30</td>
</tr>
<tr>
<td>C</td>
<td>40</td>
</tr>
<tr>
<td>Below C</td>
<td>10</td>
</tr>
</tbody>
</table>
Relative Frequency Probability is Great Fun for Teaching

- Rolling Dice
- Flipping Coins
- Drawing from Bags without Looking (i.e. Sampling)
- Sampling with M&M's
  (http://mms.com/cai/mms/faq.html#what_percent)
Empirical Probability

- Given a frequency distribution, the probability of an event, \( E \), being in a given group is

\[
P(E) = \frac{\text{frequency of the group}}{\text{total frequencies in the distribution}} = \frac{x}{n}
\]
Two-way Tables and Probability

<table>
<thead>
<tr>
<th></th>
<th>Made A</th>
<th>Made &lt; A</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>30</td>
<td>45</td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>60</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Find $P(M)$
- $P(A)$
- $P(A\text{ and } M)$
Question: How Can You Win at Wheel of Fortune?

Answer: Use Relative Frequency Probability (see handout)

What is **wrong** with the statements?

- The probability of rain today is -10%.
- The probability of rain today is 120%.
- The probability of rain or no rain today is 90%.

$$P(event) \geq 0$$
$$P(event) \leq 1$$
$$P(sample \ space) = 1$$
Let A and B be events

Complement Rule:
\[ P(A) + P(\text{not } A) = 1 \]
Set Notation

- **Union:** \( A \) or \( B \) (inclusive “or”)

- **Intersection:** \( A \) and \( B \)
Probability Rules

Union  \[ P(A \cup B) = P(A \text{ or } B) \]

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
Teaching Idea

- Venn Diagrams
- Kyle Siegrist’s Venn Diagram Applet

http://www.math.uah.edu/stat/applets/index.xml
## Two-way Tables and Probability

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<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>30</td>
<td>45</td>
<td>75</td>
</tr>
<tr>
<td>Female</td>
<td>60</td>
<td>65</td>
<td>125</td>
</tr>
<tr>
<td>Total</td>
<td>90</td>
<td>110</td>
<td>200</td>
</tr>
</tbody>
</table>

- Find $P(M)$
- $P(A)$
- $P(A \text{ and } M)$
- $P(A \text{ if } M)$
Conditional Probability

\[ P(A|B) = \text{the conditional probability of event } A \text{ happening given that event } B \text{ has happened} \]

“probability of A given B”

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)}
\]
Independence

- Events A and B are “Independent” if and only if

\[ P(A | B) = P(A) \]

- From the two-way table, is making an “A” independent from being male?
<table>
<thead>
<tr>
<th>Teaching Idea: Discovery Worksheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>❖ Basic Probability Rules (see handout)</td>
</tr>
<tr>
<td>❖ Basic Probability Rules (long version)</td>
</tr>
<tr>
<td>❖ Conditional Probability</td>
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</tbody>
</table>
Overview

- Probability Review

- Fundamentals of Counting
  - Permutations: *ordered arrangements*
  - Combinations: *unordered arrangements*

- Selected Activities
Counting Techniques

- Fundamentals of Counting
- Permutations: *ordered arrangements*
- Combinations: *unordered arrangements*
Fundamentals of Counting

Q: Jill has 9 shirts and 4 pairs of pants. How many different outfits does she have?

A:
Fundamentals of Counting

Multiplication Principle:

If there are \( a \) ways of choosing one thing, and \( b \) ways of choosing a second thing after the first is chosen, then the total number of choice patterns is:

\[ a \times b \]
Q: 3 freshman, 4 sophomores, 5 juniors, and 2 seniors are running for SGA representative. One individual will be selected from each class. How many different representative orderings are possible?

A:
Generalized Multiplication Principle:

If there are \( a \) ways of choosing one thing, \( b \) ways of choosing a second thing after the first is chosen, and \( c \) ways of choosing a third thing after the first two have been chosen...and \( z \) ways of choosing the last item after the earlier choices, then the total number of choice patterns is \( a \times b \times c \times \ldots \times z \).
Q: When I lived in Madison Co., AL, the license plates had 2 fixed numbers, 2 variable letters and 3 variable numbers. How many different license plates were possible?

A:
Fundamentals of Counting

Q: How many more license plate numbers will Madison County gain by changing to 3 letters and 2 numbers?

A:
Q: Given 6 people and 6 chairs in a line, how many seating arrangements (orderings) are possible?

A:
Q: Given 6 people and 4 chairs in a line, how many different orderings are possible?

A:
### Permutations: Ordered Arrangements

- **Permutation of** $n$ objects taken $r$ at a time:
  \[ r\text{-permutation, } P(n,r), \ nPr \]

- **Q:** Given 6 people and 5 chairs in a line, how many different orderings are possible?

- **A:**
Permutations: Ordered Arrangements

\[ nP_r = n(n-1) \cdots (n-(r-1)) \]
\[ = n(n-1) \cdots (n-r+1) \]
\[ = n(n-1) \cdots (n-r+1) \frac{(n-r)!}{(n-r)!} \]
\[ = n(n-1) \cdots (n-r+1)(n-r) \cdots (3)(2)(1) \]
\[ = \frac{n!}{(n-r)!} \]

\[ nP_r = \frac{n!}{(n-r)!} \]
Permutations: Ordered Arrangements

Q: How many different batting orders are possible for a baseball team consisting of 9 players?

A:
Q: How many different batting orders are possible for the leading four batters?

A:
Q: How many different letter arrangements can be formed using the letters TENNESSEE?

A: There are 9! permutations of the letters TENNESSEE if the letters are distinguishable.

However, 4 E’s are indistinguishable. There are 4! ways to order the E’s.
Permutations:
Indistinguishable Objects, Cont.

- 2 S’s and 2 N’s are indistinguishable. There are 2! orderings of each.

- Once all letters are ordered, there is only one place for the T.

If the E’s, N’s, & S’s are indistinguishable among themselves, then there are

\[
\frac{9!}{(4!\cdot 2!\cdot 2!)} = 3,780
\]

different orderings of T E N N E S S E E
Permutations: Indistinguishable Objects

Subsets of Indistinguishable Objects

Given $n$ objects of which
$\textcolor{red}{a}$ are alike, $\color{red}{b}$ are alike, ..., and $\color{red}{z}$ are alike

There are \( \frac{n!}{a! \cdot b! \cdots z!} \) permutations.
Combinations: Unordered Arrangements

- **Combinations:** number of different groups of size \( r \) that can be chosen from a set of \( n \) objects (order is irrelevant)

- Q: From a group of 6 people, select 4. How many different possibilities are there?

- A: There are \( 6P_4 = 360 \) different **orderings** of 4 people out of 6.

\[
6 \cdot 5 \cdot 4 \cdot 3 = 360 = \frac{n!}{(n-r)!}
\]
However the order of the chosen 4 people is irrelevant. There are 24 different orderings of 4 objects.

\[ 4 \cdot 3 \cdot 2 \cdot 1 = 24 = 4! = r! \]

Divide the total number of orderings by the number of orderings of the 4 chosen people.

\[ \frac{360}{24} = 15 \text{ different groups of 4 people.} \]
The number of ways to choose $r$ objects from a group of $n$ objects.

$C(n,r)$ or $\binom{n}{r}$, read as “$n$ choose $r$”

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]
Combinations: Unordered Arrangements

Q: From a group of 20 people, a committee of 3 is to be chosen. How many different committees are possible?

A:
Combinations:

Unordered Arrangements

Q: From a group of 5 men & 7 women, how many different committees of 2 men & 3 women can be found?

A:
Advanced web problems on permutations/combinations:
http://www.math.uah.edu/stat/comb/index.xml

The Birthday Problem
- http://www.mste.uiuc.edu/reese/birthday/intro.html (simulation applet)
- http://mathforum.org/dr.math/faq.birthdayprob.html (good details)
Overview

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- Fundamentals of Counting
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  - Combinations: unordered arrangements

- Selected Activities
Activity-based Materials for Learning Probability and Statistics

- Materials reviewed and demonstrated (simulations, discovery learning, group work)
- Overview of AP statistics
Contact Information

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