10/11/2007 MATH371 Intro. to Prob./Stats. Name: Solution
Dr. Lunsford Midterm Exam (100 Points Total)

<u>Neatly show all work on this test.</u> Answers without any work shown will not receive ***any*** credit. Clearly indicate your answers. You may use any of the following formulas (if needed) to work this test. If you use one of these formulas, clearly indicate which formula (by its number) you use. Good luck!

(1) $\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k}=(a+b)^{n}$

(2) $\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$

(3) $\sum_{n=k}^{\infty} r^{n}=\frac{r^{k}}{1-r},\ |r|<1$

(4) $\sum_{i=1}^{n} i^{2}=\frac{n(n+1)(2n+1)}{6}$

<u>Problem I.</u> Suppose P is a probability function on a sample space S and A and B are events in S such that $P(A)=0.60$, $P(B)=0.35$, and $P(A\cup B)=0.74$. Please answer the following. <u>You must show one intermediate step on each computation for full credit</u>. (4 points each, 16 points total)

a. $P(A'\cup B') = P((A\cap B)') = 1-P(A\cap B) = 1-.21 = \boxed{.79}$

$P(A\cap B)=P(A)+P(B)-P(A\cup B)$
$= .60+.35-.74 = .21$

b. Find the probability that event A or event B but not both occur.

$P(A\cup B)-P(A\cap B) = .74-.21 = \boxed{.53}$

c. Are the events A and B mutually exclusive events? Why or why not?

$P(A\cap B)\neq 0 \Rightarrow A\cap B\neq\emptyset$ (otherwise $P(A\cap B)=0$)

Thus, since $A\cap B\neq\emptyset$, then A & B are <u>not</u> mutually exclusive.

d. Are the events A and B independent events? Why or why not?

$P(A)P(B)=.60(.35)=.21=P(A\cap B)$

Since $P(A\cap B)=P(A)P(B)$ then A & B are independent.

Problem II. Suppose the discrete random variable X has probability mass function $f(x)=\dfrac{11-2x}{24}, x=1,2,3,4$. Please answer the following, being sure to show all work. (21 points total)

(a) Find $E[X]$. (5 points)

$$\rightarrow = \sum_{x=1}^{4} x\, f(x) = 1\cdot\frac{9}{24} + 2\cdot\frac{7}{24} + 3\cdot\frac{5}{24} + 4\cdot\frac{3}{24}$$

$$= \frac{50}{24} \approx 2.083$$

(b) Find $E\left[X^2\right]$. (5 points)

$$\rightarrow \sum_{x=1}^{4} x^2 \cdot f(x) = 1\cdot\frac{9}{24} + 4\cdot\frac{7}{24} + 9\cdot\frac{5}{24} + 16\cdot\frac{3}{24}$$

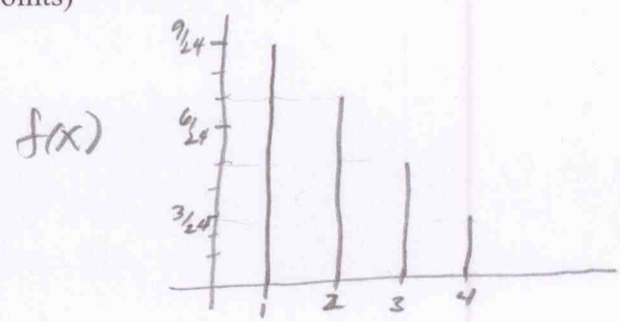
$$= \frac{130}{24} \approx 5.42$$

(c) Use the results from parts (a) *and* (b) to find $\mathrm{var}[X]$. (3 points)

$$\mathrm{var}[X] = E[X^2] - (E[X])^2 = \frac{130}{24} - \left(\frac{50}{24}\right)^2$$

$$= \frac{155}{144} \approx 1.08$$

(d) Graph the probability mass function and show μ_X on your graph. Be sure to label your axes on the graph. (5 points)



(e) What is $P(X \le 2.5)$? (3 points)

$$P(X \le 2.5) = P(X=1) + P(X=2)$$

$$= \frac{9}{24} + \frac{7}{24} = \frac{16}{24} = \boxed{\frac{2}{3}}$$

Problem III. In each of the following scenarios a random variable X is described. In each scenario give the name of the distribution of the random variable and the formula for the probability mass function of the random variable, including for which values the p.m.f is defined. Clearly indicate your answers. (4 points each, 12 total)

(a) According to the US 2000 Census, 29.5% of Virginians have a Bachelor's Degree or Higher. A Virginian is randomly chosen. If the Virginian has a Bachelor's Degree or higher let $X = 1$, otherwise let $X = 0$.

Bernoulli, $p = .295$, $f(x) = (.295)^x (.705)^{x-1}$, $x = 0, 1$

(b) Assuming the same US 2000 Census information as in part (a), 100 Virginians are randomly sampled (you may assume *with* replacement since the population is quite large). Let X be the number in the sample that *do not* have a Bachelor's Degree or higher.

Binomial, $n = 100$, $p = .705$

$f(x) = \binom{100}{x} (.705)^x (.295)^{100-x}$, $x = 0, 1, \ldots, 100$

(c) Three blind mice and seven seeing mice are in a cage. Each mouse is assigned a unique number from 1 to 10. A fair ten-sided dice is rolled and the mouse with that number is removed from the cage. Let X be the number of the mouse that was removed.

Discrete Uniform

$f(x) = \frac{1}{10}$, $x = 1, 2, \ldots, 10$

Problem IV. On a good winter day Hep Kat has a 10% chance of catching a mouse in the house, independent of what happened on previous days. During the month of December (which has 31 days), find the percent chance of each of the following events. You may assume that all of the days in December are good winter days. You must clearly indicate the meaning of any random variables you may use. (5 points each, 10 total)

(a) Hep Kat does not catch any mice.

Let X be the # of mice caught in December (assume 1 mouse/day)

$X \sim b(100, .10)$

$P(X = 0) = \binom{31}{0}(.10)^0(.90)^{31} = .038$

3.8% chance

(b) Hep Kat catches at least 3 mice.

$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2)$

$= 1 - \text{binomcdf}(31, .10, 2)$

$= .6114$

61.1% chance

Problem V. We are going to play the following game: We will take turns rolling a fair ten-sided die. I will roll first, you will roll next, then me, etc. We will stop the game when the first 1 or 8 appears on a roll. I will win if that roll is my roll, otherwise you will win. Please answer the following questions. (14 points total)

(a) What is the probability that you will win on *your first roll*? On *your second roll*? On *your third roll*? Clearly indicate your answers and please leave them in combinatorial form. (8 points)

$P(W \text{ on } 1^{st} \text{ roll}) = \frac{4}{5} \cdot \frac{1}{5}$

$P(W \text{ on } 2^{nd} \text{ roll}) = \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)$

$P(W \text{ on } 3^{rd} \text{ roll}) = \left(\frac{4}{5}\right)^5 \left(\frac{1}{5}\right)$

(b) Find the probability that you will win the game. (6 points)

$$P(\text{win}) = \sum_{i=1}^{\infty} \left(\frac{4}{5}\right)^{2i-1} \left(\frac{1}{5}\right)$$

using formula (3)

$$= \frac{1}{5}\left(\frac{4}{5}\right)^{-1} \sum_{i=1}^{\infty} \left(\frac{16}{25}\right)^{i} = \frac{1}{4}\left(\frac{\frac{16}{25}}{\frac{9}{25}}\right) = \frac{4}{9}$$

Problem VI. A mortgage company had applications for a home mortgage loan from 25 equally financially qualified applicants. Seven of these applications were from minority applicants. Because of limited resources, the mortgage company was only able to approve ten of the applications. Let the random variable X denote the number of minority candidates that were approved. Please answer the following. (8 points total)

(a) If the company was randomly choosing from the equally qualified applicants, what is the probability that at least one of the approved applications would have been from a minority applicant? Please be sure to write this probability in terms of the random variable X. (6 points)

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{18}{10}}{\binom{25}{10}} = .9866$$

(b) None of the minority applicants were approved. This particular mortgage company has been accused of discriminating against minority applicants. Based on your computation in part (a), do you think there is any evidence to support this accusation? Why or why not? (2 points)

Yes, there is evidence to support this accusation. If all applicants are equally qualified and the lender chooses randomly (i.e. without bias), there is only a 1.3% chance of not choosing any minority applications

$P(C|A_3)$ $P(C|A_1)$ $P(C|A_2$

Problem VII. Online chat rooms are dominated by the young. If we look only at adult Internet users (age 18 and over), 47% of the 18 to 29 age group chat, as do 21% of those aged 30 to 49 and just 7% of those 50 and over. Suppose that 29% of adult internet users are age 18 to 29, another 47% are 30 to 49 years old, and the remaining 24% are age 50 or older. Let C be the event that an adult internet user chats, A_1 be the event that an adult internet user is age 18 to 29, A_2 be the event that an adult internet user is age 30 to 49, and A_3 be the event an adult internet user is age 50 or older. What percent of the adult internet users who chat are in the 18 to 29 age group? Please be sure to write all probabilities you use for this computation in terms of the event names given above. (10 points total)

$P(A_1)$ $P(A_3)$ $P(A_2)$

$$P(A_1|C) = \frac{P(C|A_1)P(A_1)}{P(C)}$$

$$= \frac{P(C|A_1)P(A_1)}{P(C|A_1)P(A_1) + P(C|A_2)P(A_2) + P(C|A_3)P(A_3)}$$

$$= \frac{.47(.29)}{.47(.29) + .21(.47) + .07(.24)} = .5413$$

54% of internet users who chat are in the 18 to 29 age group.

Problem VIII. On a separate piece of paper, work ONE of the following problems. Clearly indicate which problem you want me to grade. BONUS: Work the other problem (in this case clearly indicate which problem counts towards the test and which problem is your bonus problem). (9 points)

(a) If A and B be are independent events in a sample space S then show that A and B' are also independent.

(b) If X is a discrete random variable with the discrete uniform distribution, i.e. the p.m.f. for X is $f(x) = \frac{1}{m}, x = 1, 2, ..., m$, then show that $\sigma^2 = \frac{m^2 - 1}{12}$. You may use the fact that $\mu = \frac{m+1}{2}$.