Pledge:

4/21/2010 MATH361 Calculus III Name: Solution

Dr. Lunsford Test 2 (100 Points Total)

Please show all work on this test. You may (or may not) find the following formulas useful.

$$\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin(2u) \qquad \int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin(2u) \qquad \iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

$$\int f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt \qquad \int \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot \mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_C P\,dx + Q\,dy \qquad \oint_C F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Problem I. Find the maximum rate of change of $f(x,y) = \frac{x^2}{y}$ at the point $(4,2)$ and the direction in which it occurs. Clearly indicate your answers. (8 points)

$\nabla f = \langle \frac{2x}{y}, -x^2 y^{-2} \rangle$

$\nabla f(4,2) = \langle 4, -4 \rangle$ maximum $= |\nabla f(4,2)|$ $= 4\sqrt{2}$

$\uparrow$ Direction

Problem II. Find the directional derivative of $f(x,y) = \ln(x^2+y^2)$ at the point $(2,1)$ in the direction of the vector $\mathbf{v} = \langle -1, 2 \rangle$. (8 points)

$\nabla f = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle$

$\nabla f(2,1) = \langle \frac{4}{5}, \frac{2}{5} \rangle$

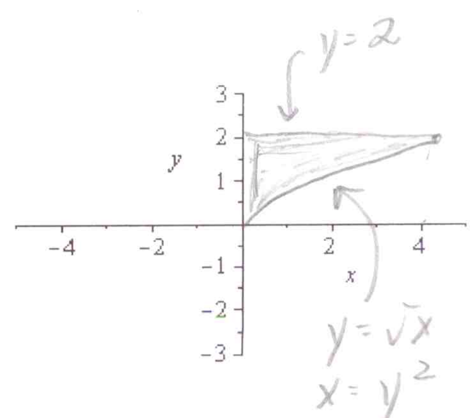
$\nabla f \cdot \frac{\vec{v}}{|\vec{v}|} = \langle \frac{4}{5}, \frac{2}{5} \rangle \cdot \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$= -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$

Problem III. Change the order of integration for the following integral. On the axes provided include a graph of the region R over which the integral is defined. DO NOT EVALUATE the integral. (10 points)

$$\int_0^4 \int_{\sqrt{x}}^2 \cos(y^2)\,dy\,dx$$

$$\int_0^2 \int_0^{y^2} \cos y^2\, dx\, dy$$



Problem IV. Set up, but DO NOT EVALUATE, an integral to find the surface area of the parametric surface given by the vector function $\mathbf{r}(u,v) = v^2\mathbf{i} - uv\mathbf{j} + u^2\mathbf{k}$ for $0 \le u \le 3$ and $-3 \le v \le 3$. (10 points)

$\vec{r}_u = \langle 0, -v, 2u \rangle$

$\vec{r}_v = \langle 2v, -u, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u^2, 4uv, 2v^2 \rangle$

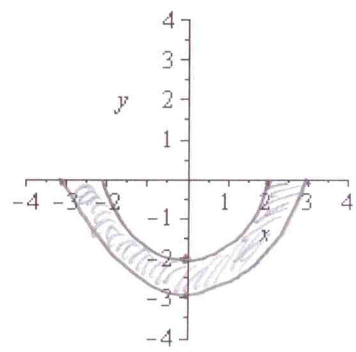
$|\vec{r}_u \times \vec{r}_v| = 2\sqrt{u^2 + 4u^2v^2 + v^2}$

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$$\int_{-3}^{3}\int_{0}^{3} 2\sqrt{u^2+4u^2v^2+v^2}\, du\, dv$$

Problem V. Change the following integral to polar coordinates: $\iint_R (x+y)\,dA$ where R is the region below the x-axis and between the graphs of $x^2+y^2=4$ and $x^2+y^2=9$. On the axes provided include a graph of the region R over which the integral is defined. DO NOT EVALUATE the integral. (10 points)

$$\int_{\pi}^{2\pi}\int_{2}^{3} (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$



Problem VI. Evaluate the line integral $\int_C y\sin(z)\,ds$ where C is the circular helix given by $x=\cos(t)$, $y=\sin(t)$ and $z=t$, $0 \le t \le 2\pi$. (10 points)

$ds = |\vec{r}'(t)|\,dt$

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_0^{2\pi} \sin(t)\sin(t)\sqrt{2}\,dt$$

$$\to \int_0^{2\pi} \sqrt{2}\sin^2 t\, dt$$

$$= \sqrt{2}\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= \sqrt{2}\left(\tfrac{1}{2}(2\pi) - 0\right)$$

$$= \sqrt{2}\pi$$